Numerical Integration

May 14, 2014

1 Integration Definition

The integration of a function $f(x)$ from $x = a$ to $x = b$ is written

$$\int_a^b f(x)dx\,.$$

This integral gives the area under the graph of f, with the area under the positive part counting as positive area, and the area under the negative part of f counting as negative area, see Fig. 1.

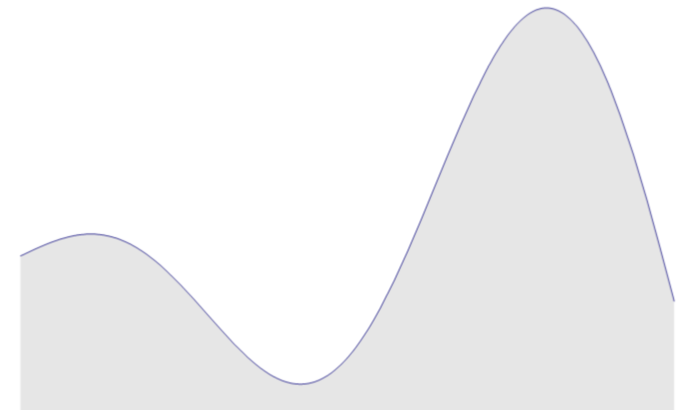

Figure 1: The area under the graph of a function.

The traditional definition of the integral is based on a numerical approximation to the area. We pick a partition $\{x_i\}$ of $[a, b]$, and in each subinterval $[x_{i-1}, x_i]$ we determine the maximum and minimum of f (for convenience we assume that these values exist),

$$mi = \min_{x \in [x_{i-1}, x_i]} f(x), \qquad Mi = \max_{x \in [x_{i-1}, x_i]} f(x),$$

for $i = 1, 2, \cdots, n$. We use these values to compute the two sums

$$\underline{I} = \sum_{i=1}^{n} mi(x_i - x_{i-1}), \qquad \overline{I} = \sum_{i=1}^{n} Mi(x_i - x_{i-1})\,.$$

To define the integral, we consider larger partitions and consider the limits of $\underline{I}$ and $\overline{I}$ as the distance between neighbouring x_is goes to zero. If those limits are the same, we say that f is integrable, and the integral is given by this limit. More precisely,

$$I = \int_a^b f(x)dx = \sup \underline{I} = \inf \overline{I},$$

where the sup and inf are taken over all partitions of the interval $[a, b]$. This process is illustrated in Fig. 2 where we see how the piecewise constant approximations become better when the rectangles become narrower.

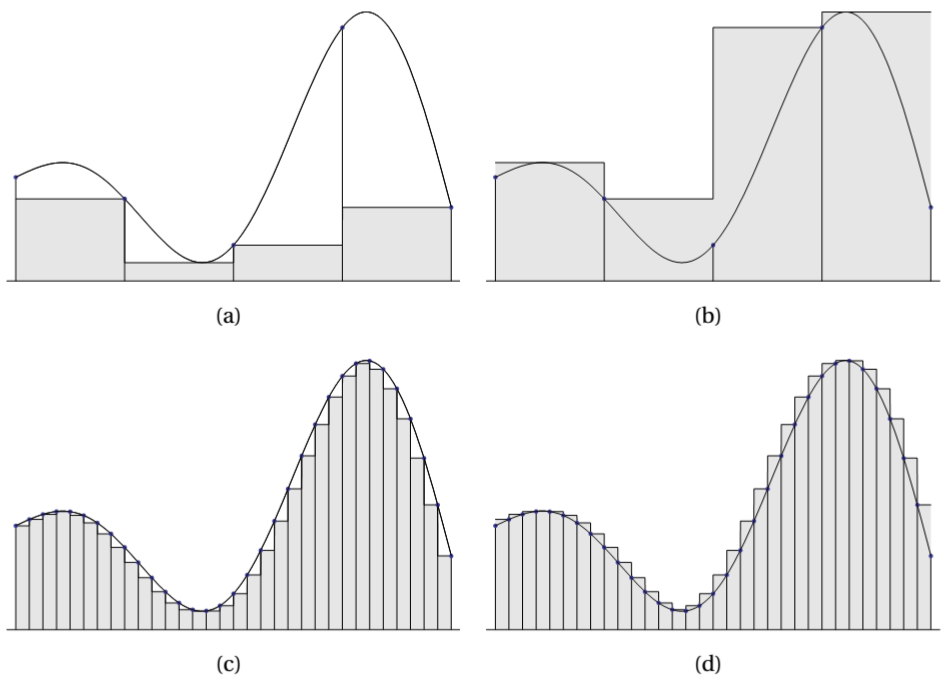

Figure 2: The definition of the integral via inscribed and circumsribed step functions.

It can be shown that the integral has a very convenient property: If we choose a point t_i in each interval $[x_{i-1}, x_i]$, then the sum

$$\tilde{I} = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

will also converge to the integral when the distance between neighbouring x_is goes to zero. If we choose t_i equal to x_{i-1} or x_i, we have a simple numerical method for computing the integral. An even better choice is the more symmetric $t_i = (x_i + x_{i-1})/2$ which leads to the approximation

$$I \approx \sum_{i=1}^{n} (f\left(x_i + x_{i-1}\right)/2)(x_i - x_{i-1})\,.$$

This is the so-called midpoint method which is shown in Fig 3 and will be studied next.

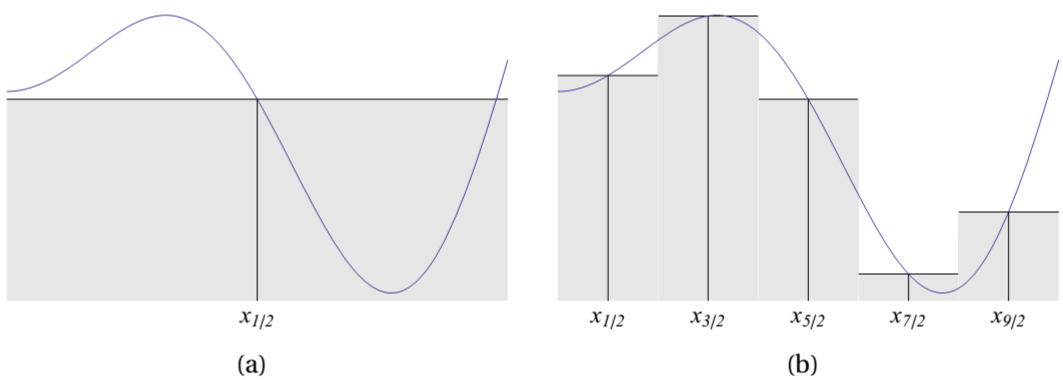

Figure 3: The midpoint rule with one subinterval (a) and five subintervals (b).

2 Midpoint Method

Algorithm:

$$\int_a^b f(x)dx \approx I_{mid}(h) = h\sum_{i=1}^{n} f(x_{i-1/2}),$$

where

$$x_{i-1/2} = (x_i + x_{i-1})/2 = a + (i - 1/2)h, \qquad \text{and}\, h = (b-a)/n$$

3 The Trapezoid Rule

The midpoint method is based on a very simple polynomial approximation to the function f to be integrated on each subinterval; we simply use a constant approximation by interpolating the function value at the middle point. We are now going to consider a natural alternative; we approximate f on each subinterval with the secant that interpolates f at both ends of the subinterval. The numerical integration under the trapezoid rule is therefore

$$\int_a^b f(x)dx \approx \frac{f(a)+f(b)}{2}(b-a)\,.$$

To get good accuracy, we will have to split $[a, b]$ into subintervals with a partition and use this approximation on each subinterval, see Fig 4. If we have a uniform partition $\{x_i\}_{i=0}^n$ with step length h, we get the approximation

$$\int_a^b f(x)dx = \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)dx \approx \sum_{i=1}^{n}\frac{f(x_{i-1})+f(x_i)}{2}h\,.$$

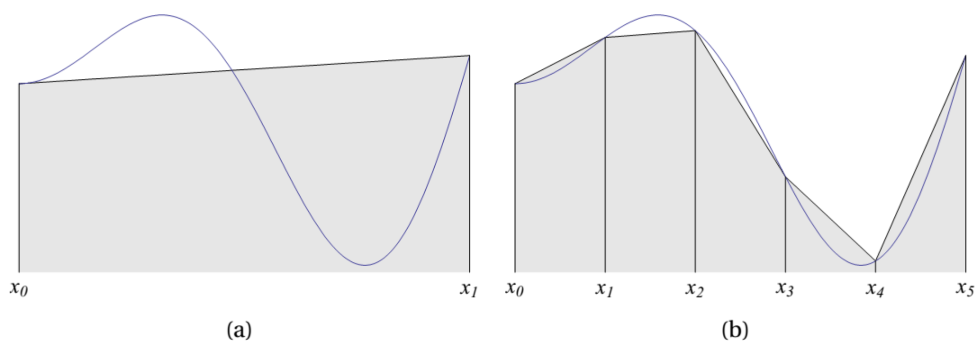

Figure 4: The trapezoid rule with one subinterval (a) and five subintervals (b).

Algorithm:

$$\int_a^b f(x)dx \approx h\left(\frac{f(a)+f(b)}{2}+\sum_{i=1}^{n-1} f(x_i)\right) .$$

4 Simpson's Rule

The final method for numerical integration that we consider is Simpson's rule. This method is based on approximating f by a parabola on each subinterval (see Fig. 5), which makes the derivation a bit more involved. The numerical integration under the Simpson's rule is therefore

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right) .$$

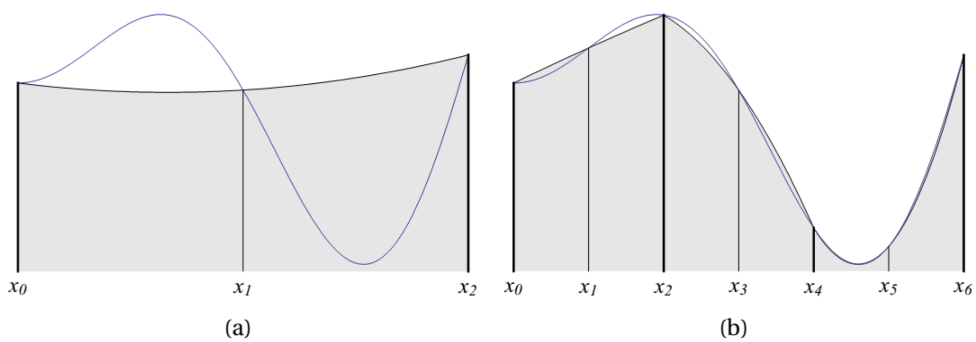

Figure 5: Simpson's rule with one subinterval (a) and three subintervals (b).

Algorithm:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_{2i}) + 4\sum_{i=1}^{n} f(x_{2i-1})\right) .$$