Root-Finding — Newton's Method

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1 Examples

Example-1: Find the root of the equation $e^{-x}-5x=0$ using Newton's method.

Solution

Since the Newton method is given using the iterative equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

using $f(x) = e^{-x} - 5x$, then $f'(x) = -e^{-x} - 5$. Starting with an initial value $x_0 = 1$, the iterations can be computed as shown in Table 1 which shows a stop at iteration no. 4 since the error is $x_4 - x_3 < 10^{-5}$ resulting in a root of $x^* = 0.16892$, see Figure 1.

Table 1: Iterations for Example-1

Iteration no.	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1} using (1)
1	$x_0 = 1$	-4.63212	-5.36788	0.13707
2	0.13707	0.18656	-5.87191	0.16884
3	0.16884	4.44036×10^{-4}	-5.84464	0.16892
4	0.16892	-2.35332×10^{-5}	-5.84458	0.16892

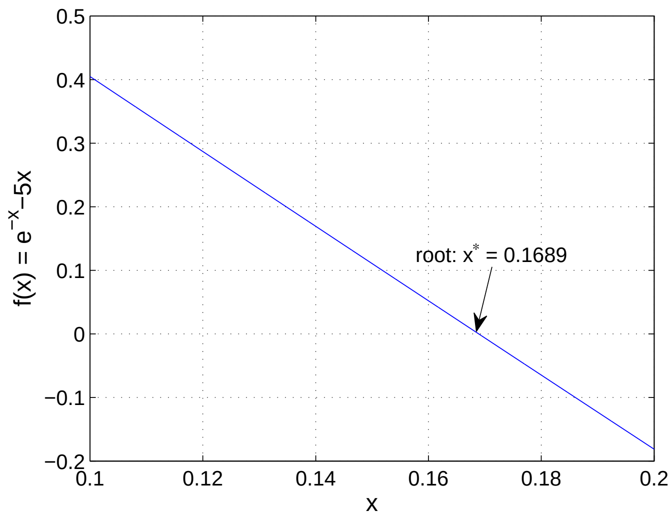

Figure 1: A plot of $f(x) = e^{-x} - 5x$ using MATLAB.

Example-2: Apply Newton's method to solve $f(x) = x^3 - 3x - 5$.

Solution

$f(x) = x^3 - 3x - 5$ and $f'(x) = 3x^2 - 3$, beginning with $x_0 = 3$, the iterates are given in Table 2 which shows a stop at iteration no. 5 since the error is $x_5 - x_4 < 10^{-5}$ resulting in a root of $x^* = 2.27902$, see Figure 2.

Table 2: Iterations for Example-2

Iteration no.	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1} using (1)
1	$x_0 = 3$	13	24	2.45833
2	2.45833	2.48165	15.13016	2.29431
3	2.29431	0.19399	12.79158	2.27914
4	2.27914	0.00153	12.58344	2.27902
5	2.27902	0.000015	12.581796	2.27902

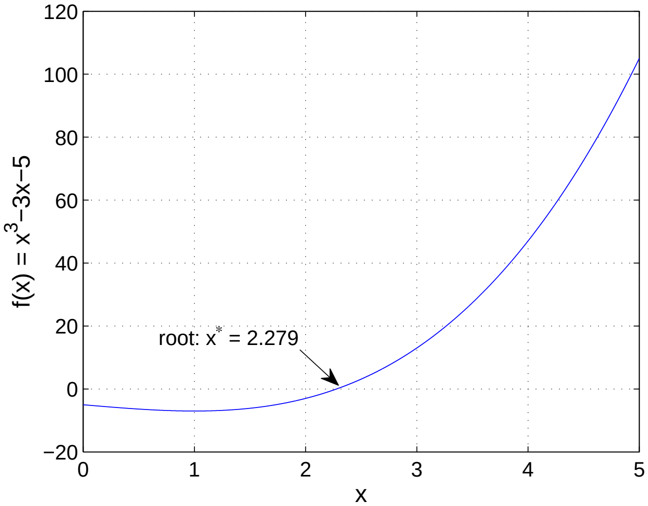

Figure 2: A plot of $f(x) = x^3 - 3x - 5$ using MATLAB.

Example-3: Apply Newton-Raphson method to find $\sqrt{2}$.

Solution

The value $\sqrt{2}$ is the solution of the equation $x^2 - 2 = 0$, i.e, the root of the function $f(x) = x^2 - 2$ with the 1st derivative $f'(x) = 2x$. Beginning with $x_0 = 2$, the iterates are given in Table 3 which shows a stop at iteration no. 5 since the error is $x_5 - x_4 < 10^{-5}$ resulting in a root of $x^* = 1.41421$.

Table 3: Iterations for Example-3

Iteration no.	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1} using (1)
1	$x_0 = 2$	2	4	1.5
2	1.5	0.25	3	1.41667
3	1.41667	0.00695	2.83334	1.41422
4	1.41422	0.000018	2.82844	1.41421
5	1.41421	-0.00001	2.82842	1.41421

Example-4: Solve $\cos x = 2x$ (x in $\cos x$ is in radians) to 5 decimal places using Newton's method.

Solution

Solving for x in the given equation is equivalent to find the root of the function $f(x) = \cos x - 2x$ where the 1st derivative is $f'(x) = -\sin x - 2$. Beginning with $x_0 = 0$, the iterates are given in Table 4 which shows a stop at iteration no. 4 since the error is $x_4 - x_3 < 10^{-5}$ resulting in a root of $x^* = 0.45018$, see Figure 3.

Table 4: Iterations for Example-4

Iteration no.	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1} using (1)
1	$x_0 = 0$	1	-2	0.5
2	0.5	-0.12242	-2.47943	0.45063
3	0.45063	-0.00109	-2.43553	0.45018
4	0.45018	8.79397×10^{-6}	-2.43513	0.45018

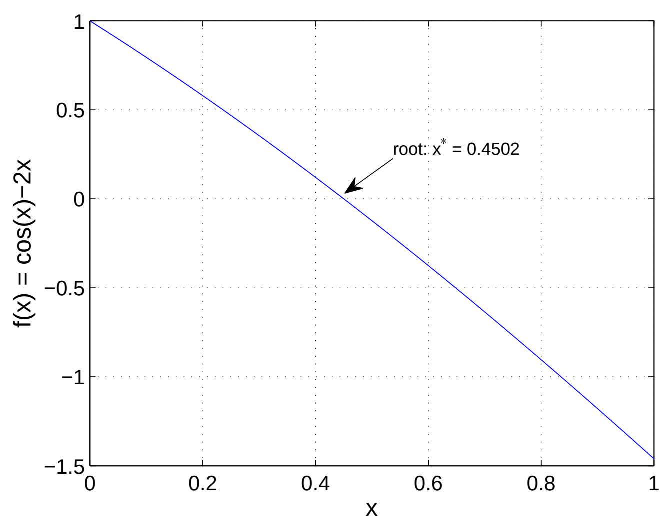

Figure 3: A plot of $f(x) = \cos x - 2x$ using MATLAB.

2 Newton-Raphson Methods Drawbacks

1. It cannot handle multiple roots.
2. It has slow convergence (compared with newer techniques).
3. The solution may diverge near a point of inflection.
4. The solution might oscillates new local minima or maxima.
5. With near-zero slope, the solution may diverge or reach a different root.

3 Algorithm

Newton's Method Algorithm

Given equation $f(x) = 0$, a predefined error ϵ, and a maximum no. of iterations N
Let the initial guess be x_0
Do

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \cdots$$

while the error $x_{n+1} - x_n < \epsilon$ or $n = N$

4 Exercises

1. Find the root of the function $y = x^3 + 4x^2 + 7$ in the vicinity of $x = -4$ correct to 5 decimal places.
2. Use Newton's Method to find the only real root of the equation $x^3 - x - 1 = 0$ correct to 5 decimal places.

3. Using Newton's method solve $x = \tan x$. Use $x_0 = 4$ and repeat the solution with $x_0 = 4.6$. Comment on the results in both cases.

4. Use the Newton-Raphson method, with 3 as starting point, to find $\sqrt{10}$.

5. Let $f(x) = x^2 - a$. Show that the Newton method leads to the recurrence $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$.

6. Newton's equation $y^3 - 2y - 5 = 0$ has a root near $y = 2$. Starting with $y_0 = 2$, compute y_1,y_2,and y_3, the next three Newton-Raphson estimates for the root.