Transforrm-Domain Representation of Signals

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Overview

Discrete-Time Fourier Transform

The discrete-time Fourier transform (DTFT) of a discrete-time signal $x(nT)$ is defined as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT} \quad (1)$$

It shows that $X(\omega)$ is a periodic function with period 2π. Thus, the frequency range of a discrete-time signal is unique over the range $(-\pi, \pi)$ or $(0, 2\pi)$.
The DTFT of $x(nT)$ can also be defined in terms of normalized frequency as

$$X(F) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j2\pi Fn} \quad (2)$$

DTFT (cont'd)

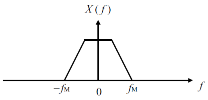
Comparing this equation with the Fourier transform of the analog $x(t)$, $X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$, the periodic sampling imposes a relationship between the independent variables t and n as $t = nT = n/f_s$. It can be shown that

$$X(F) = \frac{1}{T}\sum_{k=-\infty}^{\infty} x(f - kf_s) \quad (3)$$

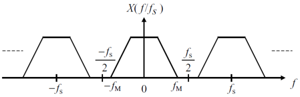
This equation states that $X(F)$ is the sum of an infinite number of $X(f)$, scaled by $1/T$, and then frequency shifted to kf_s. It also states that $X(F)$ is a periodic function with period $T = 1/f_s$.

Example

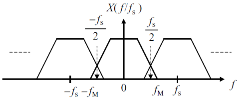
Assume that a continuous-time signal $x(t)$ is bandlimited to f_M, i.e., $|X(f)| = 0$ for $|f| \geq f_M$, where f_M is the bandwidth of signal $x(t)$. The spectrum is zero for $|f| \geq f_M$ as shown in the figure (a) below.



(a) Spectrum of an analog signal



(b) Spectrum of discrete-time signal when the sampling theorem is satisfied



(c) Spectrum of discrete-time signal when the sampling theorem is violated

Example (cont'd)

As shown in (3), sampling extends the spectrum $X(f)$ repeatedly on both sides of the f-axis. When the sampling rate f_s is greater than $2f_M$, i.e., $f_M \leq f_{s/2}$, the spectrum $X(f)$ is preserved in $X(F)$ as shown in figure (b). In this case, there is no aliasing because the spectrum of the discrete-time signal is identical (except the scaling factor $1/T$) to the spectrum of the analog signal within the frequency range $|f| \leq f_s/2$ or $|F| \leq 1$. The analog signal $x(t)$ can be recovered from the discrete-time signal $x(nT)$ by passing it through an ideal lowpass filter with bandwidth f_M and gain T . This verifies the sampling theorem.

However, if the sampling rate $f_s < 2f_M$, the shifted replicas of $X(f)$ will overlap as shown in figure (c). This phenomenon is called aliasing since the frequency components in the overlapped region are corrupted.

Discrete Fourier Transform

The DFT of a finite-duration sequence $x(n)$ of length N is defined as

$$x(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn}, \qquad k = 0, 1, \cdots, N-1 \qquad (4)$$

where $X(k)$ is the kth DFT coefficient and the upper and lower indices in the summation reflect the fact that $x(n) = 0$ outside the range $0 \leq n \leq N-1$. The DFT is equivalent to taking N samples of DTFT $X(\omega)$ over the interval $0 \leq \omega < 2\pi$ at N discrete frequencies $\omega_k = 2\pi k/N$, where $k = 0, 1, \cdots, N-1$. The spacing between two successive $X(k)$ is $2\pi/N$ rad (or f_s/N Hz).

Example

If the signal $\{x(n)\}$ is real valued and N is an even number, we can show that

$$X(0) = \sum_{n=0}^{N-1} x(n)$$

and

$$X(N/2) = \sum_{n=0}^{N-1} e^{-j\pi n} x(n) = \sum_{n=0}^{N-1} (-1)^n x(n) \, .$$

Therefore, the DFT coefficients $X(0)$ and $X(N/2)$ are real values.

$$x(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn}, \qquad k = 0, 1, \cdots, N-1$$

Another form of DFT

The DFT defined in (4) can also be written as

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0, 1, \cdots, N-1 \tag{5}$$

where

$$W_N^{kn} = e^{-j(\frac{2\pi}{N})kn} = \cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right), \qquad 0 \le k, n \le N-1\,. \tag{6}$$

The parameter W_N^{kn} is called the twiddle factors of the DFT. Because $W_N^N = e^{-j2\pi} = 1 = W_N^0$, $W_N^k, k = 0, 1, \cdots, N-1$ are the N roots of unity in clockwise direction on the unit circle.

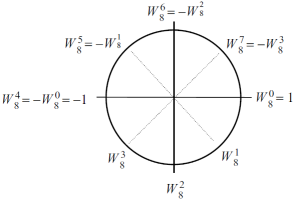
Twiddle Factors Properties

It can be shown that $W_N^{N/2} = e^{-j\pi} = -1$.
The twiddle factors have the symmetry property:

$$W_N^{k+N/2} = -W_N^k \qquad 0 \leq k \leq N/2 - 1, \tag{7}$$

and the periodicity property:

$$W_N^{k+N} = W_N^k \,. \tag{8}$$



Twiddle factors for DFT, $N = 8$

Example

Consider the finite-length signal

$$x(n) = a^n, \qquad n = 0, 1, \cdots, N-1$$

where $0 < a < 1$. The DFT of $x(n)$ is computed as

$$x(k) = \sum_{n=0}^{N-1} a^n e^{-j(2\pi k/N)n} = \sum_{n=0}^{N-1} \left(ae^{-j(2\pi k/N)}\right)^n$$

$$X(k) = \frac{1 - \left(ae^{-j(2\pi k/N)}\right)^N}{1 - ae^{-j(2\pi k/N)}} = \frac{1 - a^N}{1 - ae^{-j(2\pi k/N)}}, \quad k = 0, 1, \cdots, N-1$$

Conclusions

Concluding remarks

- The Discrete Fourier Transform (DFT) has been discussed
- As an initial step it is convenient to consider the Discrete-Time Fourier Transform (DTFT)
- Some properties and examples of the DFT have been given