Transforrm-Domain Representation of Signals: Discrete Fourier Transform (DFT)

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Overview

Common Z-Transform Pairs

signal, $x(n)$	z-transform, $X(z)$	ROC
$\delta(n)$	1	all z
$\delta(n-n_0)$	z^{-n_0}	$z \neq 0$
$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$e^{-\alpha n} u(n)$	$\frac{1}{1-e^{-\alpha} z^{-1}}$	$\|z\| > \|e^{-\alpha}\|$
$nu(n)$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\|z\| > 1$

Partial-Fraction Expansion Method

Example: Find the inverse Z-Transform of the following function using the partial-fraction expansion method:

$$X(z) = \frac{z(z-3)}{(z-2)(z-4)(z+5)}$$

Answer:

$$\begin{aligned}\frac{X(z)}{z} &= \frac{(z-3)}{(z-2)(z-4)(z+5)} = \frac{A}{(z-2)} + \frac{B}{(z-4)} + \frac{C}{(z+5)} \\ &= \frac{1}{14(z-2)} + \frac{1}{18(z-4)} - \frac{8}{63(z+5)}\end{aligned}$$

$$\begin{aligned}\therefore X(z) &= \frac{(1/14)z}{(z-2)} + \frac{(1/18)z}{(z-4)} - \frac{(8/63)z}{(z+5)} \\ \Rightarrow x(n) &= \left(\frac{1}{14}2^n + \frac{1}{18}4^n - \frac{8}{63}(-5)^n\right) u(n)\end{aligned}$$

The frequency response of a digital system can be readily obtained from its transfer function $H(z)$ by setting $z = e^{jw}$ and obtain

$$H(\omega) = H(z)\mid_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \mid_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} . \qquad (1)$$

Thus, the frequency response $H(\omega)$ of the system is obtained by evaluating the transfer function on the unit circle $|z| = \left|e^{j\omega}\right| = 1$. The digital frequency is in the range of $-\pi \leq \omega \leq \pi$.
The characteristics of the system can be described using the frequency response. In general, $H(\omega)$ is a complex-valued function expressed in polar form as

$$H(\omega) = |H(\omega)|\, e^{j\phi(\omega)},$$

where $|H(\omega)|$ is the magnitude (or amplitude) response and $\phi(\omega)$ is the phase response. The magnitude response $|H(\omega)|$ is an even function of ω, and the phase response $\phi(\omega)$ is an odd function. Thus, we only need to evaluate these functions in the frequency region $0 \leq \omega \leq \pi$. $|H(\omega)|^2$ is the squared-magnitude response, and $|H(\omega_0)|$ is the system gain at frequency ω_0.

Examples

- The moving-average filter expressed as

$$y(n) = 0.5\,[x(n) + x(n-1)]\,, \quad n \geq 0$$

is a simple first-order FIR filter. Taking the z-transform of both sides and rearranging the terms, we obtain

$$H(z) = 0.5\left(1 + z^{-1}\right)\,.$$

From(1), we have

$$H(\omega) = 0.5\left(1 + e^{-j\omega}\right) = 0.5\,(1 + \cos\omega - j\sin\omega)\,,$$

$$|H(\omega)|^2 = \{Re\,[H(\omega)]\}^2 + \{Im\,[H(\omega)]\}^2 = 0.5(1 + \cos\omega),$$

$$\phi(w) = \tan^{-1}\left\{\frac{Im\,[H(\omega)]}{Re\,[H(\omega)]}\right\} = \tan^{-1}\left(\frac{-\sin\omega}{1 + \cos\omega}\right),$$

Examples (cont'd)

since $\sin\omega = 2\sin\left(\frac{\omega}{2}\right)\cos\left(\frac{\omega}{2}\right)$ and $\cos\omega = 2\cos^2\left(\frac{\omega}{2}\right) - 1$.
Therefore, the phase response is

$$\phi(\omega) = \tan^{-1}\left[-\tan\left(\frac{\omega}{2}\right)\right] = -\frac{\omega}{2}.$$

The frequency response can be analysed using the MATLAB function:

```
[H,w]=freqz(b,a,N);
```

which returns the N-point frequency vector w and the complex frequency response vector H.

- Consider the IIR filter defined as

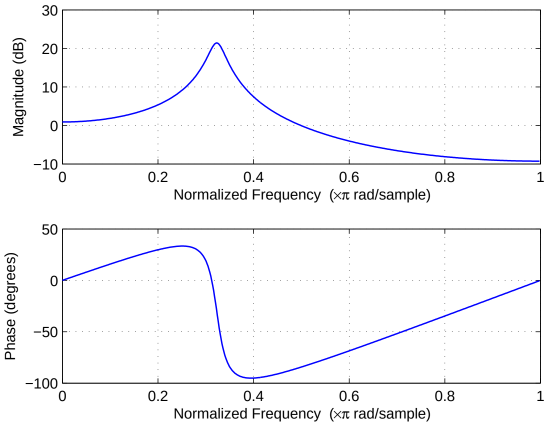
$$y(n) = x(n) + y(n-1) - 0.9y(n-2).$$

The transfer function is $H(z) = \frac{1}{1-z^{-1}+0.9z^{-2}}$

Examples (cont'd)

The MATLAB script for analysing the magnitude and phase responses of this IIR filter is listed as follows:

```
b=[1];
a=[1, -1, 0.9];
freqz(b,a);
```



To perform frequency analysis of $x(n)$,we can convert the time-domain signal into frequency domain using the z-transform, and the frequency analysis can be performed by substituting $z = e^{j\omega}$. However, $X(w)$ is a continuous function of continuous frequency ω, and it also requires an infinite number of $x(n)$ samples for calculation. Therefore, it is difficult to compute $X(\omega)$ using digital hardware.

The discrete Fourier transform (DFT) of N-point signals $\{x(0), x(1), x(2), \cdots, x(N-1)\}$ can be obtained by sampling $X(\omega)$ on the unit circle at N equally-spaced samples at frequencies $\omega_k = 2\pi k/N, k = 0, 1, \cdots, N-1$. From (1), we have

$$X(k) = X(w)\mid_{\omega=2\pi k/N} = \sum_{n=0}^{N-1} x(n) e^{-j\left(\frac{2\pi k}{N}\right)n}, k = 0, 1, \cdots, N-1,$$

where n is the time index, k is the frequency index, and $X(k)$ is the kth DFT coefficient.

- The DFT is equivalent to taking N samples of DTFT $X(\omega)$ over the interval $0 \leq \omega < 2\pi$ at N discrete frequencies $\omega_k = 2\pi k/N$, where $k = 0, 1, \cdots .N - 1$. The spacing between two successive $X(k)$ is $2\pi/N$ rad (or f_s/N Hz).
- The DFT can be manipulated to obtain a very efficient computing algorithm called the fast Fourier transform (FFT).

MATLAB provides the function `fft(x)` to compute the DFT of the signal vector x. The function `fft(x,N)` performs N-point FFT. If the length of x is less than N, then x is padded with zeros at the end. If the length of x is greater than N, function `fft(x,N)` truncates the sequence x and performs DFT of the first N samples only. DFT generates N coefficients $X(k)$ for $k = 0, 1, \cdots, N - 1$. The frequency resolution of the N-point DFT is

$$\Delta = \frac{f_s}{N}. \quad (2)$$

The frequency f_k (in Hz) corresponding to the index k can be computed by

$$f_k = k\Delta = \frac{kf_s}{N}, k = 0, 1, \cdots, N-1\,. \tag{3}$$

The Nyquist frequency ($f_s/2$) corresponds to the frequency index $k = N/2$. Since the magnitude $|X(k)|$ is an even function of k, we only need to display the spectrum for $0 \leq k \leq N/2$ (or $0 \leq \omega_k \leq \pi$).

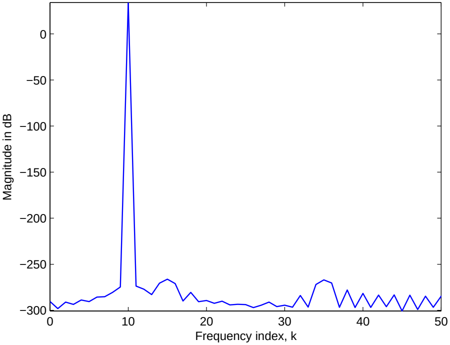
A MATLAB Example

Generate 100 samples of sinewave with $A = 1$, $f = 1$ kHz, and sampling rate of 10 kHz. Find the magnitude response of the signal and plot using MATLAB

```
N=100; f = 1000; fs = 10000;
n=[0:N-1]; k=[0:N-1];
omega=2*pi*f/fs;
xn=sin(omega*n);
Xk=fft(xn,N);                   % Perform DFT
magXk=20*log10(abs(Xk));        % Compute magnitude spectrum
plot(k, magXk);
axis([0, N/2, -inf, inf]);      % Plot from 0 to pi
xlabel('Frequency index, k');
ylabel('Magnitude in dB');
```

A MATLAB Example (cont'd)

From (2), frequency resolution is 100 Hz. The peak spectrum shown in the Figure below is located at the frequency index $k = 10$, which corresponds to 1000 Hz as indicated by (3).



Conclusion

Concluding remarks

- The Z-transform of some common signals are introduced
- Inverse Z-transform using partial-fraction expansion is given
- The frequency response of systems' transfer functions is discussed
- An introduction to DFT is studied