Transforrm-Domain Representation of Signals: The Z-Transform

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Overview

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References:

[1] S. M. Kuo, B. H. Lee, and W. Tian, Real-Time Digital Signal Processing Implementations and Applications, 2nd ed. John Wiley & Sons Ltd, 2006.

[2] S. Haykin, Adaptive Filter Theory, 3rd ed. Prentice Hall, 1996.

Z-Domain Definition

- Continuous-time systems are commonly analysed using the Laplace transform.
- For discrete-time systems, the transform corresponding to the Laplace transform is the z-transform.
- The z-transform $(\mathrm{ZT}[\cdot])$ of a digital signal, $x(n), -\infty < n < \infty$, is defined as the power series:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}, \tag{1}$$

where $X(z)$ represents the z-transform of $x(n)$, i.e., $x(n) \rightleftharpoons X(z)$.

Z-Domain Definition (cont'd)

- The variable z is a complex variable, and can be expressed in polar form as
$$z = re^{j\theta},$$
where r is the magnitude (radius) of z and θ is the angle of z.
- When $r = 1, |z| = 1$ is called the unit circle on the z-plane.
- Since the z-transform involves an infinite power series, it exists only for those values of z where the power series defined in (1) converges.
- The region on the complex z-plane in which the power series converges is called the *region of convergence*.
- For causal signals, the two-sided z-transform defined in (1) becomes a one-sided z-transform expressed as $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$.

Example

Find the z-transform of the exponential function

$$x(n) = a^n u(n)\,.$$

Answer: The z-transform can be computed as

$$X(z) = \sum_{n=-\infty}^{\infty} a^n z^{-n} u(n) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n\,.$$

Using the infinite geometric series given in Appendix A, we have

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \qquad \text{if } \left|az^{-1}\right| < 1\,.$$

The equivalent condition for convergence is $|z| > |a|$, which is the region outside the circle with radius a.

The properties of the z-transform are extremely useful for the analysis of discrete-time LTI systems. These properties are summarised as follows:

1. *Linearity (superposition)*:
   The z-transform of the sum of two sequences is the sum of the z-transforms of the individual sequences. That is,

$$\begin{aligned} \mathrm{ZT}\left[a_1 x_1(n) + a_2 x_2(n)\right] &= a_1 \mathrm{ZT}\left[x_1(n)\right] + a_2 \mathrm{ZT}\left[x_2(n)\right] \\ &= a_1 X_1(z) + a_2 X_2(z), \end{aligned}$$

   where a_1 and a_2 are constants.

2. *Time shifting:*
   The z-transform of the shifted (delayed) signal

$$Y(z) = \mathrm{ZT}\left[x(n-k)\right] = z^{-k} X(z)\,.$$

   Thus, the effect of delaying a signal by k samples is equivalent to multiplying its z-transform by a factor of z^{-k}. For example, $\mathrm{ZT}\left[x(n-1)\right] = z^{-1}X(z)$. The unit delay z^{-1} corresponds to a time shift of one sample in the time domain.

3. *Convolution:*
   Consider the signal

$$x(n) = x_1(n) \star x_2(n),$$

   we have

$$X(z) = X_1(z)X_2(z)\,.$$

   The z-transform converts the convolution in time domain to the multiplication in z domain.

The Inverse Z-Transform

The inverse z-transform is defined as

$$x(n) = \mathrm{ZT}^{-1}\left[X(z)\right] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz,$$

where C denotes the closed contour of $X(z)$ taken in a counterclockwise direction.

Several methods are available for finding the inverse z-transform: long division, *partial-fraction expansion*, and *residue method*.

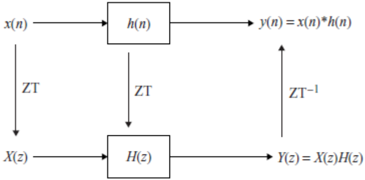
A limitation of the long-division method is that it does not lead to a closed form solution. However, it is simple and lends itself to software implementation. Both the partial-fraction-expansion and the residue methods lead to closed form solutions. The main disadvantage is the need to factorise the denominator polynomial, which is difficult if the order of $X(z)$ is high.

Definition

Consider the LTI system illustrated below. Using the convolution property, we have

$$Y(z) = X(z)H(z), \tag{2}$$

where $X(z) = \mathrm{ZT}\left[x(n)\right], Y(z) = \mathrm{ZT}\left[y(n)\right]$, and $H(z) = \mathrm{ZT}\left[h(n)\right]$. The combination of time- and frequency-domain representations of LTI system is illustrated below. This diagram shows that we can replace the time-domain convolution by the z-domain multiplication.



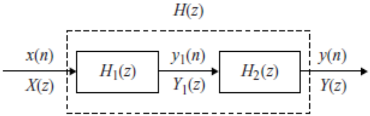
The transfer function of an LTI system is defined in terms of the system's input and output. From (2), we have $H(z) = \frac{Y(z)}{X(z)}$.

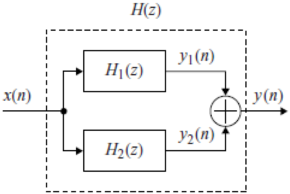
Cascade or Parallel Connection

The z-transform can be used in creating alternative filters that have exactly the same input-output behaviour. An important example is the cascade or parallel connection of two or more systems, as illustrated below. In the cascade (series) interconnection, we have

$$Y_1(z) = X(z)H_1(z) \qquad \text{and} \qquad Y(z) = Y_1(z)H_2(z)\,.$$

$$\text{Thus,} \qquad Y(z) = X(z)H_1(z)H_2(z)\,.$$





Cascade or Parallel Connection (cont'd)

Therefore, the overall transfer function of the cascade of the two systems is

$$H(z) = H_1(z)H_2(z) = H_2(z)H_1(z) .$$

The overall impulse response of the system is

$$h(n) = h_1(n) \star h_2(n) = h_2(n) \star h_1(n) .$$

Similarly, the overall impulse response and transfer function of the parallel connection of two LTI systems are given by

$$h(n) = h_1(n) + h_2(n) .$$

and

$$H(z) = H_1(z) + H_2(z)$$

If we can multiply several transfer functions to get a higher-order system,we can also factor polynomials to break down a large system into smaller sections.

An LTI Example

The LTI system with transfer function

$$H(z) = 1 - 2z^{-1} + z^{-3}$$

can be factored as

$$H(z) = \left(1 - z^{-1}\right)\left(1 - z^{-1} - z^{-2}\right) = H_1(z)H_2(z)\,.$$

Thus, the overall system $H(z)$ can be realised as the cascade of the first-order system $H_1(z) = 1 - z^{-1}$ and the second-order system $H_2(z) = 1 - z^{-1} - z^{-2}$.

Z-Transform of FIR and IIR systems

The FIR filter can be transformed to the z-transform as

$$\begin{aligned} Y(z) &= b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_{L-1} z^{-(L-1)} X(z) \\ &= \left(b_0 + b_1 z^{-1} + \cdots + b_{L-1} z^{-(L-1)} \right) X(z) \, . \end{aligned}$$

Therefore, the transfer function of the FIR filter is expressed as

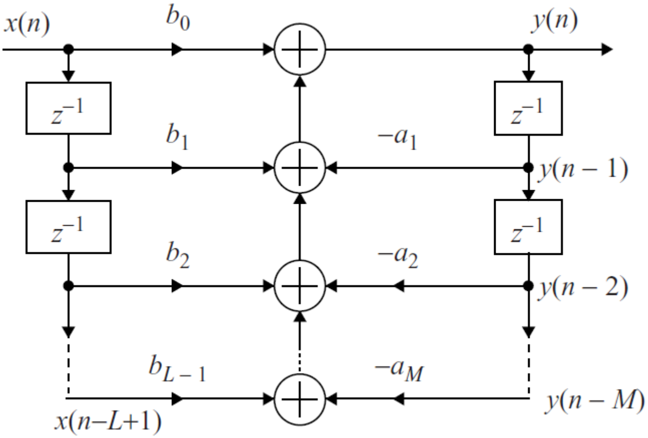
$$H(z) = b_0 + b_1 z^{-1} + \cdots + b_{L-1} z^{-(L-1)} = \sum_{l=0}^{L-1} b_l z^{-l} \, . \tag{3}$$

Similarly, taking the z-transform of both sides of the IIR filter

$$\begin{aligned} Y(z) &= b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_{L-1} z^{-(L-1)} X(z) - a_1 z^{-1} Y(z) - \cdots - a_M z^{-M} Y(z) \\ &= \left(\sum_{l=0}^{L-1} b_l z^{-l} \right) X(z) - \left(\sum_{l=1}^{M} a_m z^{-m} \right) Y(z) \, . \end{aligned}$$

$$\Rightarrow H(z) = \frac{\sum_{l=0}^{L-1} b_l z^{-l}}{1 + \sum_{m=1}^{M} a_m z^{-m}} = \frac{\sum_{l=0}^{L-1} b_l z^{-l}}{\sum_{m=0}^{M} a_m z^{-m}} \tag{4}$$

Detailed signal-flow diagram of an IIR filter



Using the geometric series

$$\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x}, \qquad x \neq 1\,.$$

The numerator and denominator polynomials of $H(z)$in (4) can be factored and expressed as the following rational function

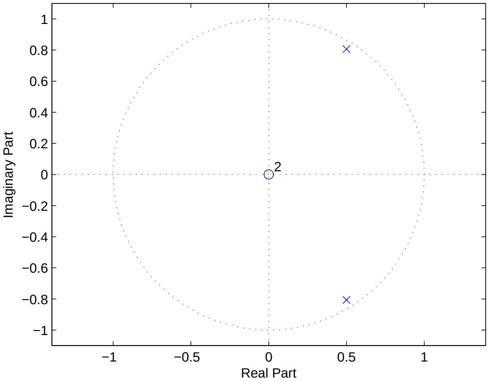
$$H(z) = b_0 \frac{\prod_{l=1}^{L-1}(z-z_l)}{\prod_{m=1}^{M}(z-p_m)} = \frac{b_0\,(z-z_1)\,(z-z_2)\cdots(z-z_{L-1})}{(z-p_1)\,(z-p_2)\cdots(z-p_M)}\,.$$

The roots of the numerator polynomial are the zeros of the transfer function $H(z)$ since they are the values of z for which $H(z)=0$. Thus, $H(z)$ given in (4) has $(L-1)$ zeros at $z = z_1, z_2, \cdots, z_{L-1}$. The roots of the denominator polynomial are the poles since they are the values of z such that $H(z)=\infty$, and there are M poles at

$z = p_1, p_2, \cdots, p_M$. The LTI system described in (4) is a pole-zero system, while the system described in (3) is an all-zero system. The pole-zero diagram provides an insight into the properties of an LTI system. To find poles and zeros of a rational function $H(z)$, we can use the MATLAB function `roots` on both the numerator and denominator polynomials. Another useful MATLAB function for analysing transfer function is `zplane(b,a)`, which displays the pole-zero diagram of $H(z)$.

Example: Consider the IIR filter with the transfer function $H(z) = \frac{1}{1 - z^{-1} + 0.9z^{-2}}$. We can plot the pole-zero diagrams using the following MATLAB script

```
b = [1];
a=[1,-1,0.9];
zplane(a,b);
```


Imaginary Part
Real Part
2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.5
0
0.5
1

Conclusion

Concluding remarks

- The Z-transform of discrete time signals is introduced
- Transfer functions of systems in the Z-domain is addressed
- Both FIR and IIR examples are given
- Poles and Zeros of systems are analysed
- System stability in terms of poles is described